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Entropy Preservation Under Markov Coding

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Using the notion of topological entropy for non-compact sets we prove that for a $C^{1+\alpha}$-map with a finite Markov partition the corresponding coding map preserves topological entropy of subsets. We also provide an example of a piecewise linear conformal repeller with a Markov coding decreasing topological entropy. These results are generalized to the notions of u-dimensions.

KEY WORDS: Markov partitions; topological entropy; symbolic dynamics.

1. INTRODUCTION

One of the most important questions in dynamical system theory is to understand the different types of qualitative behavior of the trajectories. This question is by far to complex for a complete solution. A common attempt is to restrict the studies to a given class of systems. For example the classes of Axiom A systems or of conformal repellers carry sufficient structure to lead to deep results. In particular one can understand the structure of a certain class of invariant measures, called Gibbs states. In these investigations it turned out to be useful to associate to each invariant set or invariant measure some particular characteristics. Among the characteristics which deeply reflect the dynamical behavior of the system are the notion of the Hausdorff dimension and the topological/metric entropy. In general it is quite complicated to determine the value of these quantities. But in the cases of Axiom A systems or conformal repellers the situation is much better (see ref. 10).

For these classes one often uses Markov partitions to encode the system. The coding space is a symbolic dynamical system and several useful

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tools from symbolic dynamics can be exploited. The thermodynamic formalism is one of the most fruitful tools to obtain results in this direction. It is used to prove the existence of Gibbs states (see, for example, ref. 7) and to derive their multifractal properties (see, for example, refs. 10, 12, 15, 2, and 3). The multifractal analysis presents an entire family of dimensionlike or entropylike characteristics which are linked to the systems as the whole. This means that the multifractal properties of a given Gibbs state also carry information about trajectories which are atypical with respect to this measure. For this reason they are of great interest in studying such systems. Since the multifractal analysis incooperates information about atypical points it is more subtle than the standard thermodynamic formalism. In particular one cannot neglect sets of measure zero. This makes it necessary to understand the Markov coding in a delicate way.

One of the problems of applying the coding method is that the coding map is in general not one-to-one. In particular, it is not obvious that sets of positive measure are mapped to sets of positive measure and that the values of dimensionlike characteristics are preserved. The same problem occurs if one deals with entropy. To understand the structure of the Gibbs states it is sufficient to establish the fact that the boundary of the Markov partition (see ref. 7) has zero measure for any Gibbs state. This justifies the projection of the measure from the coding space to the smooth system since the coding is one-to-one on a set of full measure. But if one is interested in dimensionlike notions one needs to understand the influence of the boundary of the partition elements, especially if one deals with atypical points. In ref. 1 Ashley *et al.* showed that in general the boundaries of the Markov partition elements can carry large entropy. This leaves the possibility that the coding looses information on large sets.

Evidently these questions arise immediately when one has to deal with other measures than Gibbs states. For example the analysis of the dimension of a non-invariant measure was used in refs. 4 and 5 to get a hand on the dimension of irregular trajectories.

Another conceptual problem arises when we want to define the topological entropy for arbitrary subsets of an invertible systems (see discussion in Section 2.3 and properties in ref. 14). There are many ways to introduce the topological entropy and these notions exhibit interesting properties for non-compact, non-invariant sets. In particular they may differ on those sets. Again it is important to control the behavior of these notions under the Markov coding.

In refs. 5 and 14 a unifying notion of dimensionlike and entropylike characteristics was introduced. This notion of u-dimensions unifies the concepts of topological entropy and Hausdorff dimension for conformal mappings and Axiom A systems. Here u is a function which essentially expresses the diameter of a cylinder set in terms of its length. The geometric structure is encoded by the coding map and the function u. In this paper we will use this unified approach to obtain results for a whole class of characteristics at once.

We will show that quite general notions of dimension and topological entropy are preserved under the coding map. The concepts and the proofs differ in some technical points for invertible and non-invertible systems. For this reason we expose these two cases separately.

Our main concept is to find Markov partitions whose elements have a comparable diameter with respect to the metric given by u. This is a generalization of the concept of a Moran cover introduced by Pesin and Weiss (ref. 12, see also ref. 10). We prove that these special partitions have a finite multiplicity independent of the diameter of the elements. This will imply that we can find codings with arbitrary small elements where the number of points with the same coding up to a given finite time is uniformly bounded. This ensures the preservation of the u-dimension.

2. AXIOM A BASIC SETS

2.1. Markov Coding

Let $f: M \to M$ be a $C^{1+\alpha}$ diffeomorphism ($\alpha > 0$) of a smooth compact finite dimensional Riemannian manifold without boundary, and $\Lambda \subset M$ a compact locally maximal hyperbolic set for f. Those sets are also called **Axiom A bassic sets**. Then, there is a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$, and constants $C > 0$ and $\lambda \in (0, 1)$ such that for each $x \in \Lambda$:

(A1) $d_x f E^s_x = E^s_{fx}$ and $d_x f E^u_x = E^u_{fx}$;

(A2) $\|d_x f^n v\| \leqslant C\lambda^n \|v\|$ for all $v \in E^s_x$ and $n \geqslant 0$;

(A3) $\|d_x f^{-n} v\| \leqslant C\lambda^n \|v\|$ for all $v \in E^u_x$ and $n \geqslant 0$.

For each point $x \in \Lambda$ there exist **local stable** and **unstable manifolds** $W^s(x)$ and $W^u(x)$, with $T_x W^s(x) = E^s_x$ and $T_x W^u(x) = E^u_x$. Moreover, there exists $\delta > 0$ such that for all $x, y \in \Lambda$ with $\rho(x, y) < \delta$, the set $W^s(x) \cap W^u(y)$ consists of a single point, which we denote by $[x, y]$, and the map

$$[\cdot, \cdot]: \{(x, y) \in \Lambda \times \Lambda : \rho(x, y) < \delta\} \to \Lambda$$

is continuous.

We may assume that $f|_\Lambda$ is topological mixing by replacing f with f^n, for some $n \in \mathbb{N}$.

A rectangle R is a nonempty closed set with $\operatorname{diam} R \leqslant \delta$, $R = \overline{\operatorname{int} R}$, and $[x, y] \in R$ whenever $x, y \in R$. The latter means that there is a Hölder continuous homeomorphism

$$\theta\colon R \to R \cap W^s(x) \times R \cap W^u(x) \tag{1}$$

for $x \in R$.

A finite cover $\mathscr{R} = \{R_1, \dots, R_l\}$ by rectangles is called a **Markov partition** for Λ if

(MP1) $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ for $i \neq j$

(MP2) For $x \in \operatorname{int} R_i \cap f^{-1}(\operatorname{int} R_j)$ we have

$$f(W^s(x) \cap R_i) \subset W^s(f(x)) \cap R_j$$

$$f(W^u(x) \cap R_i) \supset W^u(f(x)) \cap R_j.$$

Locally maximal hyperbolic sets have Markov partitions of arbitrarily small diameter. Each Markov partition has associated a two-sided subshift of finite type $\sigma|_{\Sigma_A}$ with transition matrix A, and a **coding map** $\chi\colon \Sigma_A \to \Lambda$ for the hyperbolic set, which is Hölder continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$ and $\sup\{\operatorname{card}(\chi^{-1}x)\colon x \in \Lambda\} < \infty$ (see, for example, ref. 9 for details). Let $R_{i_{-m} \cdots i_n} = \bigcap_{k=-m}^{n} f^{-k} R_{i_k}$ then $\chi(\cdots i_{-m} \cdots i_n \cdots) = \bigcap_{n, m \geqslant 0} R_{i_{-m} \cdots i_n}$. We denote the set of all those nonempty sets $R_{i_{-m} \cdots i_n}$ by $\mathscr{R}_m^n$.

We say that an element R of $\mathscr{R}_m^n$ has **length** $m(R) := n + m + 1$.

For each point $\underline{i} = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_A$, and each nonnegative integers n, m, we define the **cylinder set** $C_m^n(i_{-m} \cdots i_n) = C_m^n(\underline{i})$ as the set of points

$$\{(\cdots j_{-1}\, j_0\, j_1 \cdots) \in \Sigma_A : j_k = i_k \text{ for } i = -m, \dots, n\}.$$

Then $\chi(C_m^n(i_{-m} \cdots i_n)) = R_{i_{-m} \cdots i_n}$. We say that the above cylinder set has length $n + m + 1$.

We define the stable boundary of R_I by $\partial^s R_i := \{x \in R_i : x \notin \operatorname{int} W^u(x) \cap R_i\}$, the unstable boundary of R_i by $\partial^u R_i := \{x \in R_i : x \notin \operatorname{int} W^s(x) \cap R_i\}$ and the total boundary of R_i by $\partial R_i = \partial^s R_i \cup \partial^u R_i$. We also use the notations $\partial \mathscr{R} := \bigcup_{1 \leqslant k \leqslant l} \partial R_l$, $\partial^s \mathscr{R} := \bigcup_{1 \leqslant k \leqslant l} \partial^s R_l$ and $\partial^u \mathscr{R} := \bigcup_{1 \leqslant k \leqslant l} \partial^u R_l$. In this notation and $f(\partial^s \mathscr{R}) \subset \partial^s \mathscr{R}$ and $\partial^u \mathscr{R} \subset f(\partial^u \mathscr{R})$ (see ref. 9 for details).

We denote by $\operatorname{ind} R_{i_{-m} \cdots i_n}$ the number of rectangles $R_{i'_{-m} \cdots i'_n}$ which have nonempty intersection with $R_{i_{-m} \cdots i_n}$. We note that different rectangles of the form $R_{j_{-m} \cdots j_n}$ intersect only at their boundaries. We call the number

$$\operatorname{ind} \mathscr{R} := \sup\{\operatorname{ind} R_{i_{-m} \cdots i_n}\colon n, m \geqslant 0 \;\; \underline{i} \in \Sigma_A\}$$

the **index of the Markov partition**.

Let $\bar{u} = (u^s, u^u)\colon \Lambda \to \mathbb{R}^2$ be a Hölder continuous function with $u^s(x)$, $u^u(x) > 0$ for all $x \in \Lambda$. Given $\bar{r} = (r^s, r^u) > (0, 0)$, $x \in \Lambda$ we define a pair of numbers $\bar{n} = \bar{n}(x) = (n^s(x), n^u(x))$ by

$$n^i(x, \bar{r}) = \max\left\{k \in \mathbb{N} : \sum_{j=0}^{k} u^i(f^{\tau(i)\, j}(x)) \leqslant \frac{1}{r^i}\right\} \qquad i = s, u$$

where $\tau(s) := -1$ and $\tau(u) = 1$. Let

$$\mathscr{R}_{\bar{u}, \bar{r}} = \{R_{i_{-n^s(x, \bar{r})} \cdots i_{n^u(x, \bar{r})}} : x \in R_{i_{-n^s(x, \bar{r})} \cdots i_{n^u(x, \bar{r})}}\}.$$

Given a rectangle $R^i \in \mathscr{R}_{\bar{u}, \bar{r}}$ we call a rectangle $R^j \in \mathscr{R}_{\bar{u}, \bar{r}}$ a **$\bar{u}$, $\bar{r}$-neighbor** of R^i iff $R^i \cap R^j \neq \emptyset$. We define

$$\mathrm{ind}_{\bar{u}, \bar{r}}\, \mathscr{R}_{\bar{u}, \bar{r}} = \max_{R \in \mathscr{R}_{\bar{u}, \bar{r}}} \mathrm{card}\{R^j \in \mathscr{R}_{\bar{u}, \bar{r}} : R^j \text{ is a } \bar{u}, \bar{r}\text{-neighbor of } R\}.$$

The **$\bar{u}$-index of the Markov partition** is defined as

$$\mathrm{ind}_{\bar{u}}\, \mathscr{R} := \sup_{\bar{r} \in \mathbb{R}^+ \times \mathbb{R}^+} \{\mathrm{ind}_{\bar{u}, \bar{r}}\, \mathscr{R}_{\bar{u}, \bar{r}}\}.$$

2.2. *u*-Dimensions

Let $\bar{u} = (u^s, u^u)\colon \Lambda \to \mathbb{R}^2$ be a strictly positive Hölder continuous function. For each nonempty set $R_{i_{-m} \cdots i_n} \in \mathscr{R}^n_m$, we write

$$u^u(R_{i_{-m} \cdots i_n}) = \sup\left\{\sum_{k=0}^{n} u^u(f^k x) \colon x \in R_{i_{-m} \cdots i_n}\right\}$$

and

$$u^s(R_{i_{-m} \cdots i_n}) = \sup\left\{\sum_{k=0}^{n} u^s(f^{-k} x) \colon x \in R_{i_{-m} \cdots i_n}\right\}.$$

For each set $Z \subset \Lambda$ and each real number δ, we define

$$M(Z, \delta, \bar{u}, \mathscr{R}) = \liminf_{r \to 0}\ \inf_{\Gamma_r} \sum_{\mathbf{R} \in \Gamma_r} \exp(-\delta u^u(\mathbf{R})) - \delta u^s(\mathbf{R})), \tag{2}$$

where the infimum is taken over all finite or countable collections $\Gamma_r \subset \bigcup_{t \leqslant r} \mathscr{R}_{(u^s, u^u), (t, t)}$ that cover Z.

By a slight modification of the construction of Carathéodory dimension characteristics (see ref. 5 or ref. 10), when δ goes from $-\infty$ to $+\infty$, the quantity in Eq. (2) jumps from $+\infty$ to 0 at a unique critical value. Hence, we can define the number

$$\dim_{\bar{u}} Z = \inf\{\delta\colon M(Z, \delta, \bar{u}, \mathscr{R}) = 0\}$$
$$= \sup\{\delta\colon M(Z, \delta, \bar{u}, \mathscr{R}) = +\infty\}.$$

This number is called the **$\bar{u}$-dimension w.r.t. $\mathscr{R}$** of the set Z. It can be shown that in the case of an Axiom A basic set this number does not depend on the choice of the Markov partition (its value is the same for any generating partition, see ref. 5 or ref. 10 for details).

Remark 2.1. If $\bar{u} = (-\log a^s, \log a^u)$ where a^s and a^u are the norms of the derivative along the stable and unstable direction, respectively, of a hyperbolic horseshoe in a two-dimensional manifold, then the number $\dim_{\bar{u}} Z$ coincides with $\dim_H Z$ (see ref. 5 or ref. 10).

2.3. Topological Entropy for Non-Compact Sets

The notion of topological entropy for non-compact sets was introduced by Bowen in ref. 6. Later it was considered by Pesin and Pitskel' in ref. 11.

For invertible transformations, there is an asymmetry which occurs for non-compact or non-invariant subsets of Λ. This asymmetry arises from weighting the "future" and the "past" differently.

We propose a family of notions of topological entropy which weights the "complexity" in the "future" and in the "past."

We consider a special case of the measure considered in Eq. (2). We choose $u^s \equiv 1/p$ and $u^u \equiv 1/(1-p)$, where $0 < p < 1$.

For every set $Z \subset \Lambda$, $0 < p < 1$, and every real number δ, we set

$$N^p(Z, \delta, \mathscr{R}) = \lim_{k \to \infty} \inf_{\Gamma_k^p} \sum_{\mathbf{R} \in \Gamma_k^p} \exp(-\delta m(\mathbf{R})), \tag{4}$$

where the infimum is taken over all finite or countable collections $\Gamma_k^p \subset \bigcup_{j \geqslant k} \mathscr{R}_{(1/p,\, 1/(1-p)),\, (j^{-1}, j^{-1})}$ that cover Z and $m(\mathbf{R})$ denotes the length of the element $\mathbf{R}$. Then this becomes a special case of (2) since $m(\mathbf{R}) = u^s(\mathbf{R}) + u^u(\mathbf{R})$. We note that the sets involved in the definition of the above outer measure depend on "p percent" coordinates from the "future" and "$1-p$ percent" coordinates from the "past". We want to remark that it is also possible to include the cases $p = 0$ and $1 - p = 0$ into this family. In these cases we consider only sets that have coordinates completely in the "past" or in the "future" (see ref. 14).

The number

$$h_p(f \mid Z) = \inf\{\delta\colon N^p(Z, \delta, \mathscr{R}) = 0\} = \sup\{\delta\colon N^p(Z, \delta, \mathscr{R}) = +\infty\}$$

is called the ***p*-weighted topological entropy** of Z.

We also consider the quantity introduced in ref. 5 measuring the minimal "complexity" of all "time directions." We set

$$N^*(Z, \delta, \mathscr{R}) = \lim_{k \to \infty} \inf_{\Gamma_k^*} \sum_{\mathbf{R} \in \Gamma_k^*} \exp(-\delta m(\mathbf{R})), \tag{5}$$

where the infimum is taken over all finite or countable collections $\Gamma_k^* \subset \bigcup_{m+n>k} \mathscr{R}_m^n$ with $n, m \geqslant 0$, that cover Z.

The number

$$h_*(f \mid Z) = \inf\{\delta\colon N^*(Z, \delta, \mathscr{R}) = 0\} = \sup\{\delta\colon N^*(Z, \delta, \mathscr{R}) = +\infty\}$$

is called the **unweighted topological entropy** of Z. It is not hard to show (see ref. 5) that all the above definitions do not depend on the choice of the Markov partition and that

$$h_*(f \mid Z) \leqslant \min_{0 \leqslant p \leqslant 1} h_p(f \mid Z).$$

Moreover, if the set Z is compact and invariant all the above definitions of entropy coincide with the classically defined topological entropies (see ref. 10).

3. REPELLERS

In this section we present the modifications of the previous concepts to the case of a non-invertible map.

3.1. Markov Coding for Repellers

Let $f\colon M \to M$ be a $C^{1+\alpha}$ map of a smooth manifold, and J an f-invariant compact subset of M. We say that f is **expanding** on J and that J is a **repeller** of f if there are constants $C > 0$ and $\beta = \lambda^{-1} > 1$ such that $\|d_x f^n u\| \geqslant C\beta^n \|u\|$ for all $x \in J$, $u \in T_x M$, and $n \geqslant 1$. If in addition the derivative of f is a scalar multiple of an isometry at any point of J we call J a **conformal repeller**.

Since for expanding maps only the forward images are well-defined we have to modify all definitions accordingly. A finite cover $\mathscr{R} = \{R_1, \ldots, R_l\}$ is called a **Markov partition** for the expanding repeller J if

(MP1′) $R_i = \overline{\operatorname{int} R_i}$ for $1 \leqslant i \leqslant l$

(MP2′) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ for $i \neq j$

(MP3′) The image $f(R_i)$ of any rectangle is the union of rectangles R_j.

It is well known that repellers admit Markov partitions of arbitrarily small diameter. Again each Markov partition has associated a one-sided subshift of finite type $\sigma|\Sigma_A^+$, and a **coding map** $\chi\colon \Sigma_A^+ \to J$ for the repeller, which is Hölder continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$ and $\sup\{\operatorname{card}(\chi^{-1}x)\colon x \in J\} < \infty$ (see, for example, ref. 9 for details). Let $R_{i_0 \cdots i_n} = \{x \in M : f^k(x) \in R_{i_k} \text{ for } 0 \leqslant k \leqslant n\}$ then $\chi(i_0 \cdots i_n \cdots) = \bigcap_{n \geqslant 0} R_{i_0 \cdots i_n}$. We denote the set of all those nonempty sets $R_{i_0 \cdots i_n}$ by $\mathscr{R}_n$.

We say that an element R of $\mathscr{R}_n$ has **length** $m(R) := n+1$.

For each point $\underline{i} = (i_0 i_1 \cdots) \in \Sigma_A^+$, and each nonnegative integer n, we define the **cylinder set** $C_n(i_0 \cdots i_n) = C_n(\underline{i})$ as the set of points

$$\{j_0 j_1 \cdots) \in \Sigma_A : j_k = i_k \text{ for } i = 0,\dots, n\}.$$

Then $\chi(C_n(i_0 \cdots i_n)) = R_{i_0 \cdots i_n}$. We say that the above cylinder set has length $n+1$.

In view of property (MP3′) all the images $f^n(\partial R_i)$ are contained in the boundary $\bigcup_{k=1}^{l} \partial R_k$.

Let $R_{i_0 \cdots i_n}$, $n \geqslant 1$ be defined as above. We denote by $\operatorname{ind} R_{i_0 \cdots i_n}$ the number of rectangles $R_{i'_0 \cdots i'_n}$ which have nonempty intersection with $R_{i_0 \cdots i_n}$. We note that different rectangles of the form $R_{j_0 \cdots j_n}$ intersect only at their boundaries. As in the case of diffeomorphisms we call the number $\operatorname{ind} \mathscr{R} := \sup\{\operatorname{ind} R_{i_0 \cdots i_n}\colon n \geqslant 1\ \underline{i} \in \Sigma_A^+\}$ the **index of the Markov partition**.

Let $u\colon J \to \mathbb{R}$ be a positive Hölder continuous function. Given $r > 0$, $x \in J$ we define a number $n(x)$ by

$$n(x) = \max\left\{k \in \mathbb{N} : \sum_{j=0}^{k} u(f^j(x)) \leqslant \frac{1}{r}\right\}.$$

Let

$$\mathscr{R}_{u,r} = \{R_{i_0 \cdots i_{n(x,r)}}\colon x \in R_{i_0 \cdots i_{n(x,r)}}\}.$$

Given a rectangle $R^i \in \mathscr{R}_{u,r}$ we call a rectangle $R^j \in \mathscr{R}_{u,r}$ a ***u, r*-neighbor** of R^i iff $R^i \cap R^j \neq \emptyset$. We define

$$\operatorname{ind}_{u,r} \mathscr{R}_{u,r} = \max_{R \in \mathscr{R}_{u,r}} \operatorname{card}\{R^j \in \mathscr{R}_{u,r} : R^j \text{ is a } u, r\text{-neighbor of } R\}.$$

The ***u*-index of the Markov partition** is defined as

$$\operatorname{ind}_u \mathscr{R} := \sup_{r \in \mathbb{R}^+} \{\operatorname{ind}_{u,r} \mathscr{R}_{u,r}\}.$$

3.2. *u*-Dimensions for Repellers

Let $u\colon J \to \mathbb{R}^2$ be a strictly positive Hölder continuous function. For each nonempty set $R_{i_0 \cdots i_n} \in \mathscr{R}_n$, we write

$$u(R_{i_0 \cdots i_n}) = \sup \left\{ \sum_{k=0}^{n} u(f^k x) \colon x \in R_{i_0 \cdots i_n} \right\}.$$

For each set $Z \subset J$ and each real number δ, we define

$$M(Z, \delta, u, \mathscr{R}) = \lim_{r \to 0} \inf_{\Gamma_r} \sum_{\mathbf{R} \in \Gamma_r} \exp(-\delta u(\mathbf{R})), \tag{6}$$

where the infimum is taken over all finite or countable collections $\Gamma_r \subset \bigcup_{t \leqslant r} \mathscr{R}_{u,t}$ that cover Z.

When δ goes from $-\infty$ to $+\infty$, the quantity in Eq. (6) jumps from $+\infty$ to 0 at a unique critical value. Hence, we can define the number

$$\begin{aligned} \dim_u Z &= \inf\{\delta\colon M(Z, \delta, u, \mathscr{R}) = 0\} \\ &= \sup\{\delta\colon M(Z, \delta, u, \mathscr{R}) = +\infty\}. \end{aligned}$$

This number is called the ***u*-dimension w.r.t.** $\mathscr{R}$ of the set Z. This number does not depend on the choice of the Markov partition (its value is the same for any generating partition, see ref. 5 or ref. 10 for details).

Remark. If $u = \log a$ where a is the norm of the derivative of the map f on a conformal repeller, then the number $\dim_u Z$ coincides with $\dim_H Z$ (see ref. 5 or ref. 10). In the case of an expanding repeller the u-dimension has a close relation to the topological entropy (see Section 3.3).

3.3. Topological Entropy for Repellers

As in the previous sections concerning expanding repellers we can make the appropriate modifications of the concept of topological entropy for invertible transformations. However in the situation of an expanding repeller there is a simpler equivalent way to introduce the definition of the topological entropy for non-compact sets.

Let $Z \subset J$ be a subset. We set $u_0 \equiv 1$ and define the **topological entropy** of Z to be the number $h(f \mid Z) := \dim_{u_0} Z$.

We note that in the case of a non-invertible transformation we have only one "time direction" and weighted entropies do not make sense.

4. SYMBOLIC SPACES

In the case of the symbolic dynamical system Σ_A (invertible dynamics) or Σ_A^+ (non-invertible dynamics) we can define $\bar{u}$-dimensions (u-dimensions) and topological entropies via an obvious modification of the definitions in the previous sections. For this we consider the lifted function $\bar{u} \circ \chi\colon \Sigma_A \to \mathbb{R}^2$ ($u \circ \chi\colon \Sigma_A^+ \to \mathbb{R}$) and substitute the sets $R_{i_{-m} \cdots i_n}$ by their lifts $C_m^n(i_{-m} \cdots i_n)$. We note that the lift of a Hölder continuous function is Hölder (with possibly different exponent).

5. MAIN RESULTS

Let $C^+(\Lambda)$ be the space of positive continuous functions and $\mathscr{F}_\theta(\Lambda)$ the space of positive Hölder continuous functions on Λ with Hölder exponent θ. We denote by $\mathscr{F}_\theta^K(\Lambda)$ the family of functions in $\mathscr{F}_\theta(\Lambda)$

$$\{u \in \mathscr{F}_\theta(\Lambda) : |u(x) - u(y)| \leqslant K d(x, y)^\theta \text{ for every } x, y \in \Lambda\}.$$

The spaces $C^+(J)$, $\mathscr{F}_\theta(J)$ and $\mathscr{F}_\theta^K(J)$ are defined similarly.

Theorem 5.1. Let Λ be an Axiom A basic set for a $C^{1+\alpha}$ diffeomorphism equipped with a Markov partition $\mathscr{R}$. Then $\operatorname{ind}_{\bar{u}} \mathscr{R}\colon C(\Lambda) \times C(\Lambda) \to \mathbb{N} \cup \{\infty\}$ is uniformly bounded on each subspace $\mathscr{F}_\theta^K(\Lambda) \times \mathscr{F}_\theta^K(\Lambda)$.

Similarly,

Theorem 5.2. Let J be an expanding repeller for a $C^{1+\alpha}$ map equipped with a Markov partition $\mathscr{R}$. Then $\operatorname{ind}_u \mathscr{R}\colon C(J) \to \mathbb{N} \cup \{\infty\}$ is uniformly bounded on each subspace $\mathscr{F}_\theta^K(J)$.

Remark 5.3. In the case that a^s and a^u are the norms of the derivatives along the stable, respectively unstable, direction of a horseshoe in two dimensions Pesin and Weiss[12] proved that

$$\sup\{\operatorname{ind} R\colon r > 0 \;\; R \in \mathscr{R}_{(-\log a^s, \log a^u), (r, r)}\} =: M < \infty.$$

They called this number M the *Moran multiplicity* of $\mathscr{R}$.

The Theorems 5.1 and 5.2 have the following corollary.

Corollary 5.4. Under the assumptions of Theorem 5.1 (5.2) the map $\chi: \Sigma_A \to \Lambda$ (respectively, $\chi: \Sigma_A^+ \to J$) preserves for any subset $Z \subset \Sigma_A$ (repectively, $Z \subset \Sigma_A^+$) its $\bar{u} \circ \chi$-dimension, $\bar{u} \in \mathscr{F}_\theta(\Lambda) \times \mathscr{F}_\theta(\Lambda)$, (respectively, u-dimension, $u \in \mathscr{F}_\theta(J)$,) its p-weighted entropies, $0 \leqslant p \leqslant 1$, and its unweighted topological entropy.

6. A PIECEWISE SMOOTH COUNTEREXAMPLE

In this section we provide an example of a piecewise linear conformal mapping which exhibits a finite Markov partition, but in contrary to the smooth case the corresponding coding map decreases the topological entropy of some subsets. In particular, the index of the Markov partition with respect to the function $u \equiv 1$ is ∞.

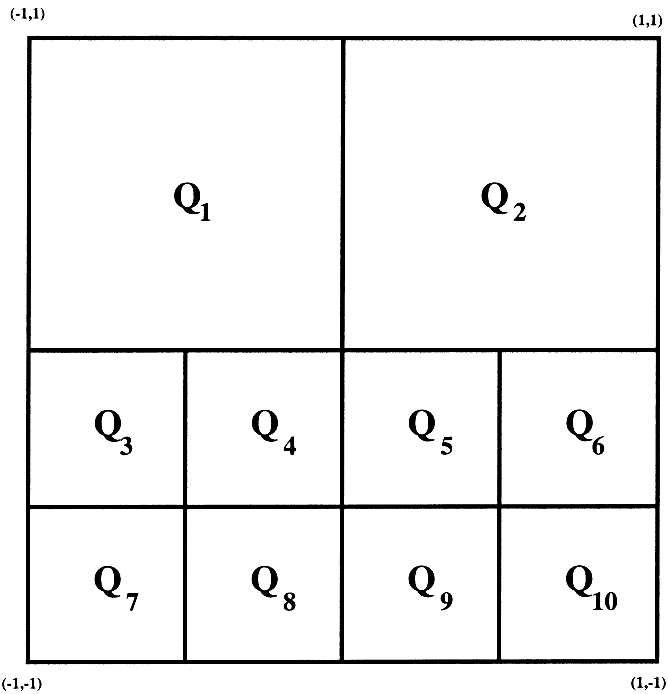

Fig. 1. Markov partition for a piecewise linear map.

Let Q be the square $[-1,1]\times[-1,1]$. We divide it into 10 squares as indicated in the figure. $Q_i=[-2+i,\ -1+i]\times[0,1]$, $i=1,2$; $Q_i=[-2.5+i/2,\ -2+i/2]\times[-0.5,0]$, $i=3,\ 4,\ 5,\ 6$, and $Q_i=[-2.5+i/2,\ -2+i/2]\times[-1,\ -0.5]$. We define a piecewise linear conformal map $f\colon Q\to Q$ by mapping each of the squares linear and conformal onto the square Q: $f_i\colon Q_i\to Q$ is defined as $f_i(x,y)=(2x+3-2i, 2y-1)$, $i=1,2$; $f_i(x,y)=(4x+9-2i, 4x+1)$, $i=3,\ 4,\ 5,\ 6$, and $f_i(x,y)=(4x+9-2i, 4x+3)$, $i=7,8,9,10$. We note that there is an ambiguity on the joint boundaries of the squares. Let $J^0:=\{x\in \underline{Q}: f^n(x)\in\bigcup_{i=1}^{10}\operatorname{int}Q_i$ for all $n\geqslant 0\}$. It is natural to call the closure $J:=\overline{J^0}$ the repeller for the mapping f. It is not hard to see that $J=Q$ and the partition $\{Q_1,\dots,Q_{10}\}$ is a Markov partition for J. We claim the following proposition.

Proposition 6.1. The coding map χ does not preserve topological entropies. In particular, the line $L_0:=[-1,1]\times\{0\}$ has topological entropy $h(f,L_0)=\log 2$ while its preimage has topological entropy $h(\sigma,\chi^{-1}(L_0))=\log 4$.

Proof. The rectangle $Q_{i_0\cdots i_n}:=\overline{\{x\in Q: f^k(x)\in\operatorname{int}Q_{i_k}, 0\leqslant k\leqslant n\}}$ has nonempty intersection with L_0 if and only if it belongs to one of the two families $\mathscr{A}:=\{Q_{i_0\cdots i_n}\colon i_0=1,2;\, i_k=7,\ 8,\ 9,\ 10,\ 1\leqslant k\leqslant n\}$ or $\mathscr{B}:=\{Q_{i_0\cdots i_n}\colon i_0=3,4,5,6\colon i_k=1,2, 1\leqslant k\leqslant n\}$. Let $\{U_j\}$ be a cover of L_0 consisting of rectangles. By choosing an efficent cover we may assume that $U_j\cap L_0\neq\varnothing$ for each j and the interiors of the rectangles U_j are mutually disjoint. Hence, $U_j\in\mathscr{A}\cup\mathscr{B}$. From the properties of the rectangles in $\mathscr{A}$ and $\mathscr{B}$ it follows that if $\{U_j\}$ contains elements from $\mathscr{A}$ then an entire binary interval of L_0 is covered by elements from $\mathscr{A}$—i.e., there is a $Q_{i_0\cdots i_n}\in\mathscr{B}$ such that $L_0\cap Q_{i_0\cdots i_n}\in\mathscr{B}$ is covered by elements $U_{j_1}, U_{j_2},\dots$ of $\mathscr{A}$. But

$$\sum_k \exp(-\alpha m(U_{j_k}))\geqslant \exp(-\alpha(n+1))\qquad \text{if}\quad \alpha\leqslant\log 4.$$

Hence, it is more efficient to use covers with elements only in $\mathscr{B}$. Then

$$\sum_j \exp(-\alpha m(U_j))\begin{cases}\leqslant \\ \geqslant\end{cases}\qquad 2^m\exp(-\alpha m)\qquad \text{if}\quad \begin{cases}\alpha\geqslant\log 2\\ \alpha\leqslant\log 2\end{cases}$$

where $m=\min m(U_j)$. Hence, the entropy of L_0 is $\log 2$.

The preimage of L_0 under χ is the set of sequences $\chi^{-1}(\mathscr{A})\cup\chi^{-1}(\mathscr{B})=\{\underline{i}\in\Sigma_A^+: i_0=1,2;\, i_k=7,8,9,10, 1\leqslant k\}\cup\{\underline{i}\in\Sigma_A^+: i_0=3,4,5,6;\, i_k=1,2, 1\leqslant k\}$. This set has topological entropy $\log 4$ since the map $g\colon \sigma\circ\chi^{-1}(\mathscr{A})\to\{0,1,2,3\}^{\mathbb{N}}$ defined by $(g(\underline{i}))_k=i_k-7$ is a topological conjugacy. ∎

7. PROOFS

7.1. Proof of Theorem 5.1

The proof has several steps. First we will prove that the number of neighboring rectangles of the same length is bounded. This proof relies on the volume lemma of Bowen and Ruelle.

Lemma 7.1. Let Λ be an Axiom A basic set of a $C^{1+\alpha}$ diffeomorphism equipped with a Markov partition $\mathscr{R}$. Then *ind* $\mathscr{R}$ is finite.

Proof. For $x \in \Lambda$, $n, m \geqslant 0$ and $r > 0$ let $B_m^n(x, r) := \{z \in M : f^k z \in B(f^k x, r) \quad -m \leqslant k \leqslant n\}$. We will show the connection of these balls to the elements of $\mathscr{R}_m^n$.

Step 1: If $\mathscr{R}$ is a Markov partition then $\mathscr{R}(p) := \bigvee_{k=-p}^{p} f^{-k}\mathscr{R}$ is a Markov partition too. Moreover, the diameter of each element of $\mathscr{R}(p)$ tends to zero if p grows. Also ind $\mathscr{R} \leqslant \max\{l^{4p+4}, \text{ind } \mathscr{R}(p)\}$ since for each pair $n, m \geqslant 0$ there are at most l^{2p+2} rectangles in $\mathscr{R}_{mp+k}^{np+k}$, $0 \leqslant k \leqslant p$. Therefore, to prove the lemma, we may assume that the diameter of the Markov partition is sufficiently small.

Step 2: Claim: If the rectangles of the Markov partition have sufficiently small diameter, then there is a number $r_0 > 0$ such that for all rectangles $R_{i_{-m} \cdots i_n}$, $m, n \geqslant 0$ there is a point $x \in R_{i_{-m} \cdots i_n}$ with

$$\min\{\max[\max_{0 \leqslant k \leqslant n} d(f^k x, \partial^s \mathscr{R}), \max_{-m \leqslant k \leqslant 0} d(f^k x, \partial^u \mathscr{R})]\} > r_0.$$

We are going to prove the statement only for the stable boundary. Since the rectangles are proper sets we can find an $r_0' > 0$ and points $x_1 \in R_1, \ldots x_l \in R_l$ such that $B(x_i, r_0') \cap \Lambda \subset R_i$; $1 \leqslant i \leqslant l$. Moreover these balls are mutually disjoint. Let $x_i^u \in \partial^s R_i \cap W^u(x)$. Then $d(x_i^u, x_k) \geqslant r_0'$ and for $y \in W^s(x) \cap R_i$ holds $W^u(y) \cap W^s(x^s u_i) \in \Lambda$. Hence, by continuity of the stable and unstable foliations $d(y, \partial^s R_i) > \frac{r_0'}{2} =: r_0$. In view of (MP2) a rectangle of the form $R_{i_{-m-n} \cdots i_0}$ completely "crosses" the rectangle R_{i_0}—i.e., $R_{i_{-m-n} \cdots i_0} \cap W^s(x_{i_0}) \neq \emptyset$. Let z be a point of this intersection. Then $d(z, \partial^s R_{i_0}) > r_0$. But $R_{j_{-m} \cdots j_n} = f^{-n}(R_{i_{-m-n} \cdots i_0})$ where $j_k = i_{k-n}$; $-m \leqslant k \leqslant n$. Therefore for $w = f^{-n} z$ holds $d(f^n w, \partial^s \mathscr{R}) = d(z, \partial^s R_{i_0}) > r_0$.

Step 3: Claim: There is an $r_1 > 0$ such that if $x \in R_{i_{-m} \cdots i_n}$ fulfills the assertion in step 2 and $y \notin R_{i_{-m} \cdots i_n}$ then $B_m^n(x, r_1) \cap B_m^n(y, r_1) = \emptyset$.

We fix $x \in R_{i_{-m} \cdots i_n}$ and $y \notin R_{i_{-m} \cdots i_n}$. Then at least one of the points $v^s = W^s(x) \cap W^u(y)$ and $v^u = W^u(x) \cap W^s(y)$ is not in $R_{i_{-m} \cdots i_n}$. Without loss of generality let it be v^u. Then there is a point $w \in W^u(x) \cap \partial^s R_{i_{-m} \cdots i_n}$ with $d(f^k x, f^k w) < 2d(f^k x, f^k y)$; $-m \leqslant k \leqslant n$. The latter inequality holds because of the continuity of the invariant foliations and since $f^k x$ and $f^k y$ are close for $-m \leqslant k \leqslant n$. Let us assume that $z \in B^n_m(x, r) \cap B^n_m(y, r)$ for some $r > 0$. Then

$$d(f^k x, f^k w) < 2d(f^k x, f^k y) \leqslant 2d(f^k z, f^k x) + 2d(f^k z, f^k y) \leqslant 4r$$

for $-m \leqslant k \leqslant n$. Since $f^k w \in \partial^s \mathscr{R}$; $k \geqslant 0$ this inequality is impossible for $4r \leqslant r_0$. So we can choose $r_1 := r_0/4$.

Step 4: Claim: There is a constant $C_1 > 1$ such that if $x \in R_{i_{-m} \cdots i_n}$ then $R_{i_{-m} \cdots i_n} \subset B^n_m(x, C_1 r_1)$.

Let $C_1 := \max_{1 \leqslant k \leqslant l} \operatorname{diam}(R_k)\, r_1^{-1}$ and $x \in R_{i_{-m} \cdots i_n}$. Then

$$\begin{aligned} R_{i_{-m} \cdots i_n} &= \{z \in \Lambda : f^k z \in R_{i_k},\ -m \leqslant k \leqslant n\} \\ &\subset \{z \in \Lambda : d(f^k z, f^k x) \leqslant C_1 r_1,\ -m \leqslant k \leqslant n\} \\ &= B^n_m(x, C_1 r_1). \end{aligned}$$

Step 5: Claim: If $R_{i_{-m} \cdots i_n} \cap R_{j_{-m} \cdots j_n} \neq \varnothing$ and $x \in R_{i_{-m} \cdots i_n}$, $y \in R_{j_{-m} \cdots j_n}$ then $B^n_m(x, 3C_1 r_1) \supset B^n_m(y, r_1)$.

Let $z \in B^n_m(y, r_1)$ and $w \in R_{i_{-m} \cdots i_n} \cap R_{j_{-m} \cdots j_n}$. Then by step 4

$$d(f^k z, f^k x) \leqslant d(f^k z, f^k y) + d(f^k y, f^k w) + d(f^k w, f^k x) \leqslant r_1 + 2C_1 r_1$$

for $-m \leqslant k \leqslant n$. Hence, $z \in B^n_m(x, 3C_1 r_1)$.

Step 6: Now we are ready to prove the lemma. By step 1 we may assume without loss of generality that $3C_1 r_1 < \varepsilon_0$ where ε_0 is from the volume lemma of Bowen and Ruelle formulated as Theorem A.1 in the Appendix. Let $R_{i_{-m} \cdots i_n}$ be a rectangle. Then all rectangles $R_{j_{-m} \cdots j_n}$ which have nonempty intersection with $R_{i_{-m} \cdots i_n}$ are contained in the ball $B^n_m(x, 3C_1 r_1)$ for some $x \in R_{i_{-m} \cdots i_n}$. To each of these rectangles $R_{j_{-m} \cdots j_n}$ there is associated a Bowen ball $B^n_m(y_j, r_1)$ with $y_j \in R_{j_{-m} \cdots j_n}$. In view of the claim of step 4 all these Bowen balls are contained in $B^n_m(x, 3C_1 r_1)$ and by step 3 can be chosen to be mutually disjoint. Hence their number does not exceed the ratio of their volumes. The volume lemma of Bowen and Ruelle (Theorem A.1 in the Appendix) implies

$$\operatorname{ind} R_{i_{-m} \cdots i_n} \leqslant \frac{\operatorname{vol}(B^n_m(x, 3C_1 r_1))}{\sup_{y \in B^n_m(x, 3C_1 r_1)} \operatorname{vol}(B^n_m(y, r_1))} \leqslant V(3C_1 r_1, r_1).$$

Since the rectangle was arbitrarily chosen the assertion of the lemma follows. ∎

Now we show that the distortion of the functions $n^s(x, r) = n^s_{u^s}(x, r)$ and $n^u(x, r) = n^u_{u^u}(x, r)$ is bounded on a Bowen ball of radius less then ε_0 where ε_0 is the constant from Theorem A.1.

Lemma 7.2. Let Λ be a basic set of an Axiom A $C^{1+\alpha}$ diffeomorphism equipped with a Markov partition $\mathscr{R}$ and $g \in \mathscr{F}^K_\theta$ a strictly positive Hölder continuous function. For $\varepsilon_0 > \varepsilon > 0$ there is a constant $N = N(\varepsilon)$ such that for $r > 0$, $m, k \geqslant 0$ and $y \in B^k_m(x, \varepsilon) \cap \Lambda$

$$|n^s_g(x, r) - n^s_g(y, r)| < N(\varepsilon) \qquad \text{and} \qquad |n^u_g(x, r) - n^u_g(y, r)| < N(\varepsilon).$$

Proof. We are going to prove the first of the two inequalities. The other is proved similarly.

Let $y \in B^k_0(x, \varepsilon) \cap \Lambda$. By the local product property there is a point $z \in \Lambda$ with $W^u(y) \cap W^s(x) = \{z\}$ and $\max\{d(x, z), d(y, z)\} < C'\varepsilon$ for some $C' > 0$ independent of ε and k. This and (A2) and (A3) yield that $d(f^j x, f^j z) \leqslant C'C\lambda^j\varepsilon$ and $d(f^j y, f^j z) \leqslant C'C\lambda^{n-j}\varepsilon$ for $0 \leqslant j \leqslant k$. By the Hölder continuity of g we have that

$$\begin{aligned}|g(f^j x) - g(f^j y)| &\leqslant |g(f^j x) - g(f^j z)| + |g(f^j y) - g(f^j z)|\\ &\leqslant K(C'C(\lambda^j\varepsilon + \lambda^{n-j}\varepsilon))^\theta\end{aligned}$$

for $0 \leqslant j \leqslant k$. Therefore

$$\begin{aligned}\left|\sum_{j=0}^{k} g(f^j x) - \sum_{j=0}^{k} g(f^j y)\right| &\leqslant \sum_{j=0}^{k} K(C'C(\lambda^j\varepsilon + \lambda^{n-j}\varepsilon))^\theta\\ &\leqslant K(C'C)^\theta \frac{1}{1-(\lambda\varepsilon)^\theta} =: K'.\end{aligned}$$

Hence, $|n^s_g(x, r) - n^s_g(y, r)| \leqslant \frac{K'}{\min_{x \in \Lambda} g(x)}$. Since $B^k_m(x, \varepsilon) \subset B^k_0(x, \varepsilon)$ for all $k, m \in \mathbb{N}$ the above inequality implies the desired result. ∎

Let$\bar{u} = (u^s, u^u) \in \mathscr{F}^K_\theta \times \mathscr{F}^K_\theta(\Lambda)$,$\bar{r} = (r^s, r^u) > (0, 0)andx \in R_{i_{-n^s(x,\bar{r})} \cdots i_{n^u(x,\bar{r})}}$. Lemma 7.1 says that the cardinality of rectangles $R_{j_{-n^s(x,\bar{r})} \cdots j_{n^u(x,\bar{r})}}$ that intersect $R_{i_{-n^s(x,\bar{r})} \cdots i_{n^u(x,\bar{r})}}$ is bounded by the finite number ind $\mathscr{R}$. As in step 4 in the proof of Lemma 7.1 we may assume that $3C_1 r_1 < \varepsilon_0$ then repeating part of step 5 we see that all these neighboring rectangles $R_{j_{-n^s(x,\bar{r})} \cdots j_{n^u(x,\bar{r})}}$ are contained in $B^{n^u(x,\bar{r})}_{n^s(x,\bar{r})}(x, 3C_1 r_1)$. By Lemma 7.2 each of these rectangles $R_{j_{-n^s(x,\bar{r})} \cdots j_{n^u(x,\bar{r})}}$ contains at most $l^{2N(3C_1 r_1)}$ rectangles from $\mathscr{R}_{\bar{u},\bar{r}}$. Therefore,

$$\operatorname{ind}_{\bar{u},\bar{r}} \mathscr{R} \leqslant l^{2N(3C_1 r_1)} \operatorname{ind} \mathscr{R} < \infty.$$

This completes the proof of Theorem 5.1.

7.2. Proof of Theorem 5.2

The proof in the case of repellers can be derived by obvious modifications of the proof of Theorem 5.1. The proof of the analogon of Lemma 7.1 can be considerably simplified if one notes that the Markov property implies that $f^n R_{i_0 \cdots i_n} = R_{i_0}$.

7.3. Proof of Corollary 5.4

We present the proof in the case of $\bar{u}$-dimensions. The proofs of the other statements need only obvious modifications.

Let $A \in \Sigma_A$ and $Z = \chi(Z)$.We are going to show that for $\bar{u} \in \mathscr{F}^K_\theta(\Lambda) \times \mathscr{F}^K_\theta(\Lambda)$ the equality $\dim_{\bar{u}} Z = \dim_{\bar{u} \circ \chi} A$ holds.

For $r > 0$, $\varepsilon > 0$ sufficiently small let $\Gamma_r \subset \bigcup_{t \leqslant r} \mathscr{R}_{(u^s, u^u), (t, t)}$ be a finite or countable cover of Z. To each $R \in \Gamma_r \cap \mathscr{R}_{(u^s, u^u), (t, t)}$ we can associate at most $\operatorname{ind}_{\bar{u}} \mathscr{R}$ rectangles $R' \in \mathscr{R}_{(u^s, u^u), (t, t)}$ that intersect R. We also note that for $R' \in \mathscr{R}_{(u^s, u^u), (t, t)}$

$$|u^s(R') - u^s(R)| < \max_{x \in \Lambda} u^s(x) \qquad \text{and} \qquad |u^u(R') - u^u(R)| < \max_{x \in \Lambda} u^u(x)$$

holds. Hence to the cover Γ_r of Z we can associate a cover $\hat{\Gamma}_r$ of A with

$$\sum_{C \in \hat{\Gamma}_r} \exp(-\delta u^u \circ \chi(C) - \delta u^s \circ \chi(C))$$

$$\leqslant \operatorname{ind}_{\bar{u}} \mathscr{R} \exp(\max_{x, y \in \Lambda} u^s(x) + u^u(y)) \sum_{\mathbf{R} \in \Gamma_r} \exp(-\delta u^u(\mathbf{R})) - \delta u^s(\mathbf{R})).$$

This ensures that $\dim_{\bar{u}} Z \geqslant \dim_{\bar{u} \circ \chi} A$. The reverse inequality is obvious.

APPENDIX

The proof of the main results was based on the following important fact of Bowen and Ruelle[8] (see also ref. 7).

Theorem A.1 [Bowen and Ruelle]. Let Λ be a basic set for an Axiom A $C^{1+\alpha}$-diffeomorphism. There is a $\varepsilon_0 > 0$ and a function $V: (0, \varepsilon_0) \times (0, \varepsilon_0) \to [1, \infty)$ such that for $0 < \varepsilon, \delta < \varepsilon_0$, all nonnegative integers n and m, $x \in \Lambda$ and $y \in B^n_m(x, \varepsilon)$

$$V(\varepsilon, \delta)^{-1} \operatorname{vol}(B^n_m(y, \delta)) \leqslant \operatorname{vol}(B^n_m(x, \varepsilon)) \leqslant V(\varepsilon, \delta) \operatorname{vol}(B^n_m(y, \delta))$$

holds.

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REFERENCES

1. J. Ashley, B. Kitchens, and M. Stafford, Boundaries of Markov partitions, *Trans. Amer. Math. Soc.* **333**(1):177–201 (1992).
2. L. Barreira, Ya. Pesin, and J. Schmeling, On a general concept of multifractality: multifractal spectra for dimensions, entropies, and Lyapunov exponents. Multifractal rigidity, *Chaos* **7**(1):27–38 (1997).
3. L. Barreira, Ya. Pesin, and J. Schmeling, Multifractal spectra and multifractal rigidity for horseshoes, *J. Dynam. Contr. Syst.* **3**(1):33–49 (1997).
4. L. Barreira and J. Schmeling, Invariant sets with zero measure and full Hausdorff dimension, *Electronic Research Announc. Amer. Math. Soc.* **3**:114–118 (1997).
5. L. Barreira and J. Schmeling, Sets of "non-typical points have full topological entropy and full Hausdorff dimension, *Isr. J. Math.* **116** (2000).
6. R. Bowen, Topological entropy for noncompact sets, *Trans. Amer. Math. Soc.* **184**: 125–136 (1973).
7. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Springer Lecture Notes in Mathematics, Vol. 470 (Springer Verlag, 1975).
8. R. Bowen and D. Ruelle, The ergodic theory of Axiom A flows, *Invent. Math.* **29**:181–202 (1975).
9. A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and Its Applications, Vol. 54 (Cambridge University Press, 1995).
10. Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics (Chicago University Press, 1997).
11. Ya. Pesin and B. Pitskel', Topological pressure and the variational principle for noncompact sets, *Functional Anal. Appl.* **18**(4):307–318 (1984).
12. Ya. Pesin and H. Weiss, A multifractal analysis of Gibbs measures for conformal expanding maps and Markov Moran geometric constructions, *J. Statist. Phys.* **86**(1/2): 233–275 (1997).
13. J. Schmeling, On the completeness of multifractal spectra, *Ergodic Theory and Dynam. Systems* **19**:1–22 (1999).
14. J. Schmeling, Time weighted entropies, *Coll. Math.* **84–85**(1/2):265–278 (2000).
15. H. Weiss, The Lyapunov and dimension spectra of equilibrium measures for conformal expanding maps, preprint, 1996.